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Dyson's theorem for curves

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ABSTRACT

Let K be a number field and X_1 and X_2 two smooth projective curves defined over it. In this paper we prove an analogue of the Dyson theorem for the product $X_1 \times X_2$. If $X_i = \mathbb{P}_1$ we find the classical Dyson theorem. In general, it will imply a self contained and easy proof of Siegel theorem on integral points on hyperbolic curves and it will give some insight on effectiveness. This proof is new and avoids the use of Roth and Mordell–Weil theorems, the theory of Linear Forms in Logarithms and the Schmidt subspace theorem.

1. Introduction

After the proof of the Mordell conjecture by Faltings (the first proof is in [Fa1], but [Fa2,B2,Vo2, Vo3] are nearer to the spirit of this paper), most of the *qualitative* results in the diophantine approximation of algebraic divisors *by rational points* over curves are solved.

Historically, the first concluding result is the Siegel's theorem: An affine hyperbolic curve contains only finitely many S-integral points; we know that we cannot suppose less on the *geometry* of the involved curve: $\mathbb{A}^1$ and $\mathbb{G}_m$ have, as soon as the field is sufficiently big, infinitely many integral points.

After a long and interesting story of partial results (Liouville, Thue, Siegel, Dyson, Gelfand, ...), Roth proved that, if α is an algebraic number then, for every $\kappa > 2$, the equation

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$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{|q|^{\kappa}}$$

admits only finitely many solutions $\frac{p}{q} \in \mathbb{Q}$. Here again, by Dirichlet's theorem, we know that, for $\kappa = 2$, the equation may have infinitely many solutions.

Eventually, the already quoted theorem of Faltings closes the story: a compact hyperbolic curve contains only finitely many rational points.

It is a fact that, from a *quantitative* point of view, we are still very far from a satisfactory answer (up to the very interesting partial results in [B1,B3,BVV,BC]): In each of the three problems quoted above we are not able to give an upper bound for the heights of the searched solution. And, even worst, we are not able to say if *there is* any solutions to each of these problems.

Let us have a closer look to the Siegel's theorem: the modern proof of it relies upon the Roth's theorem and on the Mordell–Weil's theorem or on the theory of the Linear Forms in Logarithms and again on the Mordell–Weil's theorem; recently, a new proof, based on the Schmidt's subspace theorem has been given [CZ]. Consequently, if one tries to find an effective proofs by refining the existing proof, one will crash into the problems of effectiveness in Roth's theorem and in the computation of a basis for the Mordell–Weil group of the Jacobian (problem which seems easier but not yet completely solved) or in the effectiveness in Schmidt's theorem. Nevertheless some very important cases of effective Siegel's theorem are given in [Bi]. So, at a first glance, an effective version of Siegel's theorem will be consequence of the solutions of other problems, which seems to be even more difficult. This is very unsatisfactory, also because a strong effective version of it will imply a version of the *abc*-conjecture [Su].

In this paper we prove a theorem in the spirit of the Dyson's theorem [B1] over the product of two curves. It will easily imply Siegel's theorem. Up to standard facts in algebraic geometry and in the theory of heights, the theorem is self contained and essentially elementary. Consequently it release Siegel's theorem from other big theorems. In this way Siegel's theorem becomes a result which is completely independent and, perhaps an effective version of it can be studied on its own.

We now give a qualitative statement of the main theorem of this paper; for a precise statement, cf. Section 2.

Let K be a number field, let $L_1, \ldots, L_r$ be finite extensions of K and $n := \max\{[L_i \cdot L_j : K]\}$ and denote by A the K-algebra $\bigoplus L_i$. Let X_1 and X_2 be smooth projective curves over K and $D_i = \mathrm{Spec}(A) \to X_i$, be effective geometrically reduced divisors on X_i; note that the D_i's may have different degrees. Let H_i be a line bundle of degree one over X_i and $h_{H_i}(\cdot)$ height functions associated to H_i. Finally, let S be a finite set of places of K and for every $v \in S$ let $\lambda_{D_i,v}(\cdot)$ be Weil functions associated to D_i and v.

1.1. Theorem. *Let ϑ_1, ϑ_2 and ϵ be three rational numbers such that $\vartheta_1 \cdot \vartheta_2 \geqslant 2n + \epsilon$ and $\vartheta_i \geqslant 1$. Let $\varphi : S \to [0, 1]$ be a function such that $\sum_{v \in S} \varphi(v) = 1$. Then the set of rational points $(P, Q) \in X_1(K) \times X_2(K)$ such that for every $v \in S$,*

$$\lambda_{D_1,v}(P) > \varphi(v) \cdot \vartheta_1 \cdot h_{H_1}(P)$$

and

$$\lambda_{D_2,v}(Q) > \varphi(v) \cdot \vartheta_2 \cdot h_{H_2}(Q)$$

is contained in a proper closed subset whose irreducible components are either fibers or points.

If we apply the theorem to $\mathbb{P}_1 \times \mathbb{P}_1$ and $\vartheta_1 = \vartheta_2 = \sqrt{2n} + \epsilon$ we reobtain the classical theorem of Dyson (cf. [B1]):

1.2. Corollary. *Let α be an algebraic number of degree n over $\mathbb{Q}$. Then there are only finitely many $\frac{p}{q} \in \mathbb{Q}$ such that*

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{q^{\sqrt{2n}+\epsilon}}.$$

If we apply the theorem to $C \times C$ where C is an arbitrary curve, D a reduced divisor on it, we obtain the following generalization.

1.3. Corollary. *Let C be a smooth projective curve over a number field K and M be a line bundle of degree one on it; let D be a reduced divisor of degree n over C then for all $p \in C(K)$ we have*

$$\lambda_{D,S}(p) \leqslant (\sqrt{2n} + \epsilon)h_M(p) + O(1).$$

The involved constant is not effective.

Corollary 1.3 easily implies Siegel's theorem on S-integral points. We first recall the definition of integral points: let C be a smooth projective curve defined over a number field K. Let D be an effective reduced divisor on C. Suppose that we fixed a logarithmic height function $h_D(\cdot)$ with respect to D. Let S be a finite set of places of K and $\lambda_{D,S}(\cdot)$ be a Weil function associated to S and D (cf. Section 2 for definitions and references). Let $I \subset C(K)$ be a set of rational points. The set I is said to be *integral with respect to D and S* (or (D, S)-integral) if there exists a constant C such that, for every point $P \in I$,

$$\left|h_D(P) - \lambda_{D,S}(P)\right| \leqslant C$$

(for short, we will write $\lambda_{D,S}(P) = h_D(P) + O(1)$).

1.4. Corollary *(Siegel theorem). Let K be a number field and S be a finite set of places of it. Let C be a smooth projective curve of genus g defined over a number field K. Let D be a reduced effective divisor on C different from zero. Suppose that $2g - 2 + \deg(D) > 0$. Then every set of (D, S)-integral points is finite.*

Proof. Fix a line bundle M of degree one on C. For every positive number ϵ, standard properties of heights (cf. for instance [HS]) give the existence of a constant A such that $\deg(D)h_M(\cdot) \leqslant (1+\epsilon)h_D(\cdot) + A$. Suppose that $\deg(D) \geqslant 3$. In this case $2g - 2 + \deg(D) > 0$ independently on the genus. Let I be a set of (D, S)-integral points. By definition $h_D(P) = \lambda_{S,D}(P) + O(1)$. Fix ϵ_1 very small and apply 1.3; we obtain, for every $P \in I$,

$$\deg(D)h_M(P) \leqslant (1+\epsilon)h_D(P) = (1+\epsilon)\lambda_{S,D}(P) + 0(1) \leqslant (1+\epsilon)\bigl(\sqrt{2\deg(D)} + \epsilon_1\bigr)h_M(P) + O(1).$$

If ϵ and ϵ_1 are sufficiently small, we have that $\deg(D) - (1+\epsilon)(\sqrt{2\deg(D)} + \epsilon_1) \geqslant 0$; consequently the height, with respect to M, of points P in I is bounded independently on P. From this we conclude this case.

Suppose that D is arbitrary. In this case $g \geqslant 1$. Take an étale covering $f : C' \to C$ of degree bigger then three. Then $\deg(f^*(D)) \geqslant 3$. By the theorem of Chevalley and Weil [Se, Theorem 4.2] there is a finite extension K' of K such that $f^{-1}(C(K)) \subset C'(K')$. Apply the previous case to C', $f^*(D)$ and $I' := f^{-1}(I)$ and conclude. □

Using Roth theorem and the weak Mordell–Weil theorem one obtains, if $g > 0$,

$$\lambda_{D,S}(p) \leqslant \epsilon h_M(p) + O(1);$$

which is much stronger then 1.3 (but, it implies the same qualitative result on integral points). Nevertheless, as already said, the proof we propose here is much simpler and its ineffectiveness is essentially self contained: it does not depend on other theorems.

A remark on the language and the methods used: In this paper we decided to use the language of arithmetic geometry à la Grothendieck and the Arakelov geometry; although this needs a little bit of background, which nowadays is (or should be) standard, this language allows to better understand and compute the involved constants and to understand their nature. It is our opinion that, algebro geometric and Arakelov methods, being more intrinsic and conceptual, are more adapted to understand the strategy and the ideas of a proof in diophantine geometry. In any case, in the paper we tried to recall the background in Arakelov geometry needed to understand it. For an introduction to the Arakelov geometry used in this paper cf. [MB] or the more general [BGS]. A very fast introduction to the Arakelov geometry of arithmetic surfaces is in [Ga].

Before we start the proof we would like to give a very informal argument that roughly explain the strategy of the proof.

Suppose that $D = 0 \subset \mathbb{P}^1$ is the divisor of a single point. Fix local coordinates (z_1, z_2) around $(0, 0)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi : \tilde{X} \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blow-up of $(0, 0)$ and let E be the exceptional divisor.

Suppose that we have a couple of points $(p_1, p_2) \subset \mathbb{P}^1 \times \mathbb{P}^1(K)$ such that $\lambda_{0,\infty}(p_i) \gg h_{\mathcal{O}(1)}(p_i)$ (the involved constants are not important in this argument). In particular we may suppose that, in the Euclidean topology, p_i is very near to 0. We want to prove there are only finitely many couples of such points.

Observe that $\lambda_{0,\infty}(p_i) = -\log \frac{|z_i(p_i)|}{\sqrt{1+|z_i(p_i)|^2}} \sim -\log |z_i(p_i)|$.

Since the exceptional divisor E is locally defined by z_1 (or z_2), a local computation gives $h_{\mathcal{O}(E)}(p_1, p_2) \gg h_{\mathcal{O}(1,1)}(p_1, p_2)$.

In the sequel we denote by h_i the real number $h_{\mathcal{O}(1)}(p_i)$.

Suppose that we can find constants A_i such that for an infinite sequence of positive integers d's there is a section $f \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(\frac{d}{h_1}, \frac{d}{h_2}))$ with integral coefficients (observe that $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d_1, d_2))$ is the space of polynomials with bidegree (d_i, d_2)) such that

- $\sup\{\|f\|_{FS}(z_1, z_2)\} \leqslant A_1^{\frac{d}{h_i+h_2}}$.
- $\operatorname{div}(f)$ vanishes with order at least $m := \frac{dA_2}{h_1+h_2}$ on $(0, 0)$.
- f *do not vanishes in* (p_1, p_2).

Then the strict transform of $\operatorname{div}(f)$ give rise to a section $\tilde{f} \in H^0(\tilde{X}, \pi^*(\mathcal{O}(\frac{d}{h_1}, \frac{d}{h_2})) - mE)$.

Since $\tilde{f}(p_1, p_2) \neq 0$, we find

$$h_{\pi^*(\mathcal{O}(\frac{d}{h_1}, \frac{d}{h_2}))(-mE)}(p_1, p_2) \geqslant -\frac{d}{h_1 + h_2} \log(A_1).$$

Thus, since $h_{\mathcal{O}(E)}(p_1, p_2) \gg h_1 + h_2$,

$$\frac{d}{h_1} \cdot h_1 + \frac{d}{h_2} \cdot h_2 - \frac{dA_2}{h_1 + h_2} \cdot (h_1 + h_2) \geqslant -\frac{d}{h_1 + h_2} \log(A_1).$$

And from this we deduce that

$$2 - A_2 \geqslant -\frac{\log(A_1)}{h_1 + h_2};$$

consequently p_1 and p_2 must have bounded height.

The general proof need to construct such a section and prove that it do not vanish on the point. In general one cannot work with the ideal (z_1, z_2) (in our example we blow up this ideal) but one consider a more complicated ideal (introduced in Section 3) which depend on the constants involved in the inequality between Weil functions and heights we are assuming. To construct the section with small norm one uses the Siegel lemma and it will not exist if we assume a too strong inequality; the inequalities supposed in Theorem 1.1 allow to construct such a section. One cannot prove that the section do not vanish on the point, thus one prove that, under suitable conditions on the heights, the section has a small order of vanishing on it (this is the more geometrical part of the proof): this is done in Section 4. This is also the part of the proof which is not effective. Thus one take a suitable "derivative of the section" to produce a section which do not vanish on the point and then the conclusion is essentially the one explained above.

One should notice that almost all the proofs in diophantine approximation follow this strategy (for instance, in one take only one factor, one obtain the Liouville inequality).

2. Statement of the main theorem and notations

Let K be a number field and O_K be its ring of integers. We will denote by M_K the set of (finite and infinite) places of K. Let M_∞ be the set of infinite places of K. Let S be a finite subset of M_K. We will denote by O_S the ring of S-integers of K. For every $v \in M_K$ let K_v be the completion of K at the place v, O_v be the local ring of v and k_v be its residue field. For every scheme $X \to \mathrm{Spec}(O_K)$ we will denote by X_v (resp. x_{O_v}, resp. X_K), the base change of it from $\mathrm{Spec}(O_K)$ to $\mathrm{Spec}(K_v)$ (resp. to $\mathrm{Spec}(O_v)$, resp. to $\mathrm{Spec}(K)$). Similarly, if L is an extension of K, we will denote by O_L the ring of integers of L, by X_L the base change of X to $\mathrm{Spec}(L)$, etc.

Let $L_1, \dots, L_r$ be finite extensions of K and O_{L_i} be the ring of integers of L_i. We will denote by A the O_K-algebra $\bigoplus O_{L_i}$.

We will denote by $\overline{K}$ the algebraic closure of K.

Let $X \to \mathrm{Spec}(O_K)$ be an O_K-scheme. An hermitian vector bundle $\overline{E}$ of rank r over X is a couple $(E, \langle\cdot,\cdot\rangle_\sigma)_{\sigma\in M_\infty}$ where

- E is a vector bundle of rank r over X.
- for every infinite place σ, the vector bundle E_σ is a holomorphic vector bundle over the $\mathbb{C}$-scheme X_σ; then $\langle\cdot,\cdot\rangle_\sigma$ is a C^∞ metric on E_σ (and if $\tau = \overline{\sigma}$, the metric on E_τ is the complex conjugate of the metric on E_σ).

If M is an hermitian vector bundle of rank one, we will call it *hermitian line bundle*. If M is an hermitian line bundle over $\mathrm{Spec}(O_K)$ we will define its Arakelov degree by the following formula: Let $s \in M \setminus \{0\}$; then

$$\widehat{\deg}(M) := \log\bigl(\mathrm{Card}(M/s \cdot O_K)\bigr) - \sum_{\sigma\in M_\infty} \log \|s\|_\sigma .$$

This formula is well defined because of the product formula (cf. for instance [Se]).

If $\overline{E}$ is an arbitrary hermitian vector bundle over $\mathrm{Spec}(O_K)$ then the line bundle $\bigwedge^{\max} E$ is canonically equipped with an hermitian metric; consequently we can define the hermitian line bundle $\bigwedge^{\max} \overline{E}$. We then define $\widehat{\deg}(\overline{E}) := \deg(\bigwedge^{\max}(\overline{E}))$.

Suppose that $\overline{E}_1$ and $\overline{E}_2$ are hermitian vector bundles over $\mathrm{Spec}(O_K)$ and $f : E_1 \to E_2$ is a linear map. Then, for every infinite place σ, f induces a linear map $f_\sigma : (E_1)_\sigma \to (E_2)_\sigma$. Let $\|f_\sigma\|_\sigma$ be the norm of it. Then we define $\|f\| := \sup\{\|f_\sigma\|_\sigma\}_{\sigma\in M_\infty}$.

More generally: Suppose that $X \to \mathrm{Spec}(O_K)$ is an arithmetic scheme and $\overline{E}$ is an hermitian vector bundle over it. Suppose that, for every $\sigma \in M_\infty$ the complex variety $X_\sigma(\mathbb{C})$ is projective and smooth and that we fixed a smooth hermitian metric on it. Under these conditions the O_K-module $H^0(X, E)$ has a natural structure of hermitian O_K-module: indeed, for every $\sigma \in M_\infty$, the complex vector space $H^0(X, E)_\sigma$ is equipped with the L^2 hermitian metric induced by the metric on $X_\sigma(\mathbb{C})$ and on E_σ. For every infinite place σ, the complex vector space $H^0(X, E)_\sigma$ is naturally equipped with the sup norm: $\|f\|_{\sup,\sigma} := \sup_{x\in X_\sigma(\mathbb{C})}\{\|f\|(x)\}$. The L^2 and sup norms are comparable (as explained for instance in [Bo]); consequently we can work with the norm we prefer.

Let $f_1 : \mathcal{X}_1 \to B := \mathrm{Spec}(O_K)$ and $f_2 : \mathcal{X}_2 \to B$ be two regular, semistable arithmetic surfaces over O_K. Let $\Delta_i \hookrightarrow \mathcal{X}_i \times_B \mathcal{X}_i$ $(i = 1, 2)$ be the diagonal divisor. The divisor Δ_i is, a priori, just a Weil divisor (the scheme $\mathcal{X}_i \times \mathcal{X}_i$ may be not regular); let $\mathcal{X}_i \tilde{\times} \mathcal{X}_i$ be the blow-up of it along Δ_i and $\tilde{\Delta}_i$ be the exceptional divisor.

For every infinite place σ, we fix a symmetric hermitian structure on the line bundle $(\mathcal{O}(\tilde{\Delta}_i))_\sigma$ $(i = 1, 2)$. Let $\sigma \in M_K$ be an infinite place and $P \in (\mathcal{X}_i)_\sigma(\mathbb{C})$; denoting by $\iota_P : (\mathcal{X}_i)_\sigma(\mathbb{C}) \to (\mathcal{X}_i \times \mathcal{X}_i)_\sigma(\mathbb{C})$ the embedding deduced from the map $\iota_P(x) := (x, P)$, we have a canonical isomorphism $\iota_P^* \mathcal{O}(\Delta) \simeq \mathcal{O}(P)$. For every place σ and $P \in (\mathcal{X}_i)_\sigma(\mathbb{C})$, we put on $\mathcal{O}(P)$ the metric obtained taking the pull back metric via ι_p. As a consequence, for every divisor D of $\mathcal{X}_i$, the line bundle $\mathcal{O}(D)$ is equipped with a canonical metric (depending only on the choices made until now).

Let D be an effective divisor on $(\mathcal{X}_i)_K$. For every finite set of places $S \in M_K$ we can choose a canonical representative for the Weil function $\lambda_{D,S}(\cdot)$ in the following way: First of all we take the schematic closure of D on $\mathcal{X}_i$; this will be a Cartier divisor over $\mathcal{X}_i$.

- Suppose that $S := \sigma$ is an infinite place; let $\mathbb{I}_D$ be the canonical section of $(\mathcal{O}(D))_\sigma$. Let $\|\cdot\|(\cdot)$ be the metric on $\mathcal{O}(D)_\sigma$ defined above; then we define, for every $x \in (\mathcal{X}_i)_\sigma(\mathbb{C}) \setminus \{D\}$:

$$\lambda_{D,\sigma}(x) := -\log \|\mathbb{I}_D\|(x).$$

- Suppose that $S := v$ is a finite place. Since D and $(\mathcal{X}_i)_v$ are generic fibers of their models over $\mathrm{Spec}(O_v)$, as explained in [D], the line bundle $(\mathcal{O}(D))_v$ over the K_v-scheme $(\mathcal{X}_i)_v$ is equipped with a v-adic norm; consequently the Weil function $\lambda_{D,v}(\cdot)$ is defined similarly.
- If S is arbitrary, then $\lambda_{D,S}(\cdot)$ is defined as sum if local terms as explained for instance in [HS, Chapter B8].

The choice of a metric on the diagonal induces a metric on the relative dualizing sheaf $\omega_{\mathcal{X}_i/B}$; we fix such a metric; remark that, by construction, the adjunction formula holds: for every section $P : B \to \mathcal{X}_i$ we have a canonical isomorphism

$$\omega_{\mathcal{X}_i/B}|_P \simeq \mathcal{O}(-P)|_P \tag{2.1.1}$$

of *hermitian* line bundles on B. For a general reference on this cf. [MB]. For a reference on Weil functions cf. [HS].

For every hermitian line bundle $\overline{M} := (M; \|\cdot\|)$ over $\mathcal{X}_i$ we can define a height function

$$h_{\overline{M}}(\cdot) : (\mathcal{X}_i)_K(\overline{K}) \to \mathbb{R}$$

in the following way:

Let $P \in \mathcal{X}_i(\overline{K})$. It is defined over a finite extension L. Let $(\mathcal{X}_i)_{O_L} \to \mathrm{Spec}(O_L)$ be the minimal regular model of $(\mathcal{X}_i)_L$. The point P corresponds to a section $\mathcal{P} : \mathrm{Spec}(O_L) \to (\mathcal{X}_i)_{O_L}$; we define

$$h_M(P) := \frac{1}{[L : \mathbb{Q}]} \cdot \deg\bigl(\mathcal{P}^*(\overline{M})\bigr).$$

An hermitian line bundle $\overline{M}$ on $\mathcal{X}_i$ is said to *be arithmetically ample* if its degree on the projective curve X_K is positive and $h_M(\cdot) > 0$.

Fix an arithmetically ample hermitian line bundles $(M_i, \|\cdot\|_{M_i})$ on $\mathcal{X}_i$ of generic degree one.

We will denote by $(\cdot;\cdot)$ the Arakelov intersection pairing on each of the $\mathcal{X}_i$ as defined for instance in [BGS] or [MB].

If D is an effective reduced divisor over $\mathcal{X}_i$; write $D := \sum D_j$ where each D_j is an irreducible divisor. Define the following three numbers associated to it:

– Let L_j be an extension of K where D_j splits as sum of points: if $f_j : (\mathcal{X}_i)_{O_{L_j}} \to \mathcal{X}_j$ is the minimal regular model of the base change of $\mathcal{X}_i$ to $\mathrm{Spec}(O_{L_j})$, then $f_j^*(D_j) = \sum_h P_{hj} + V$; where P_{hj} are sections and V is a vertical divisor: Then we define

$$S(D) := \max_{h,j}\left\{-\frac{1}{[L_j:\mathbb{Q}]}\cdot\big(\mathcal{O}(P_{hj});\mathcal{O}(P_{hj})\big);1\right\};$$
$$H(D) := \max_{h,j}\big\{h_{M_i}(P_{ij});1\big\};$$

and

$$T(D) := S(D)\cdot H(D).$$

Observe that, by formula (2.1.1), we have that $-(\mathcal{O}(P_{hj});\mathcal{O}(P_{hj})) = (\omega_{\mathcal{X}_i/O_L};\mathcal{O}(P_{hj}))$.

We eventually fix a positive integer and three positive rational numbers ϑ_1, ϑ_2 and ϵ such that

$$\vartheta_1\cdot\vartheta_2 \geqslant 2n+\epsilon.$$

The main theorem of this paper is the following generalization of Dyson's theorem.

2.2. Theorem. *Under the hypotheses above there exist two effectively computable constants R_1 and R_2, depending only on the $\mathcal{X}_i$, the hermitian line bundles M_i, the metrics on the diagonals, the ϑ_i and the constant ϵ, for which the following holds*:

Let $L_1,\ldots,L_r$ be finite extensions of K; denote by n the number $n := \max\{[L_i\cdot L_j:K]\}$, by O_{L_i} the ring of integers of L_i and by $\mathcal{A}$ the O_K-scheme $\mathcal{A} := \mathrm{Spec}(\bigoplus O_{L_i})$. Let $\varphi: S\to[0,1]$ be a function such that $\sum_{v\in S}\varphi(v)=1$.

Let

$$D_i : \mathcal{A}\to\mathcal{X}_i$$

be reduced effective divisors over $\mathcal{X}_i$ ($i=1,2$).

If $(P,Q)\in\mathcal{X}_1(K)\times\mathcal{X}_2(K)$ is a couple of rational points such that

(a) $h_{M_1}(P) \geqslant R_1\cdot T(D_1)\cdot T(D_2)$,
(b) *for every $v\in S$,*

$$\lambda_{D_1,v}(P) > \varphi(v)\cdot\vartheta_1\cdot h_{M_1}(P) \quad \textit{and} \quad \lambda_{D_2,v}(Q) > \varphi(v)\cdot\vartheta_2\cdot h_{M_2}(Q);$$

then

$$h_{M_2}(Q) \leqslant R_2\cdot T(D_1)\cdot T(D_2)\cdot h_{M_1}(P).$$

This will easily imply the qualitative theorem and its corollaries.

In the following sections we will introduce the tools we need for the proof of 2.2, we will give it in the last section.

3. Small sections

Let L be a finite extension of K of degree n and O_L its ring of integers. Denote by B_L the scheme $\mathrm{Spec}(O_L)$.

Let $\mathcal{L}$ be a line bundle over $B := \mathrm{Spec}(O_K)$; we will denote by $\mathcal{O}[\mathcal{L}]$ the O_K-algebra $\mathrm{Sym}(\bigoplus \mathcal{L}^{\otimes n})$ and by $\mathcal{O}[\![\mathcal{L}]\!]$ the O_K-algebra $\prod \mathcal{L}^{\otimes n}$ with the multiplicative structure given by $(a_n)\cdot(b_n) := (c_n)$ where $c_n := \sum_{i+j=n} a_i \otimes b_j$ (if $\mathcal{L}$ is the trivial line bundle $\mathcal{O}_B$ then $\mathcal{O}[\![\mathcal{O}_B]\!]$ is the usual ring of power series $O_K[\![X]\!]$). If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two line bundles we define $\mathcal{O}[\mathcal{L}_1, \mathcal{L}_2]$ and $\mathcal{O}[\![\mathcal{L}_1, \mathcal{L}_2]\!]$ in a similar way.

Let $f_{\mathcal{L}} : \mathbb{V}(\mathcal{L}) \to B$ be the affine B-scheme $\mathrm{Spec}(\mathcal{O}[\mathcal{L}])$ then it is easy to verify that:

(a) there is a canonical isomorphism $f^*(\mathcal{L}) \simeq \Omega^1_{\mathbb{V}(\mathcal{L})/B}$;
(b) if $\mathbf{0} : B \to \mathbb{V}(\mathcal{L})$ is the canonical section, there is a canonical isomorphism $\widehat{\mathbb{V}(\mathcal{L})}_{\mathbf{0}} \simeq \mathrm{Spf}(\mathcal{O}[\![\mathcal{L}]\!])$.

Suppose that $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ are hermitian line bundles. Let $\sigma \in M_\infty$. For every positive integer n, the complex vector space $\bigoplus_{a+b=n}(\mathcal{L}_1^{\otimes a} \otimes \mathcal{L}_2^{\otimes b})_\sigma$ has a natural structure of hermitian vector space. Consequently also $(\mathcal{O}[\mathcal{L}_1, \mathcal{L}_2])_\sigma = \bigoplus_{n\geqslant 0} \bigoplus_{a+b=n}(\mathcal{L}_1^{\otimes a} \otimes \mathcal{L}_2^{\otimes b})_\sigma$ has a natural structure of hermitian vector space. Let $J \subset \mathcal{O}[\mathcal{L}_1, \mathcal{L}_2]$ be an ideal; since $(\mathcal{O}[\mathcal{L}_1, \mathcal{L}_2])_\sigma$ is direct sum of finite dimensional hermitian vector space, we can find an orthonormal basis $\mathcal{B}_\sigma$ of $(\mathcal{O}[\mathcal{L}_1, \mathcal{L}_2])_\sigma$ such that $\mathcal{B}_\sigma$ is disjoint union of $\mathcal{B}_1$ and $\mathcal{B}_2$ with $\mathcal{B}_1$ orthonormal basis of J_σ. Consequently the vector space $(\mathcal{O}[\mathcal{L}_1, \mathcal{L}_2]/J)_\sigma$ is canonically (via the projection) isomorphic to $J_\sigma^\perp$, thus it is equipped with the structure of hermitian vector space. Moreover, suppose that $J_1 \subset J_2$, then the metric induced by the canonical projection $\mathcal{O}[\mathcal{L}_1, \mathcal{L}_2]/J_2 \to \mathcal{O}[\mathcal{L}_1, \mathcal{L}_2]/J_1$ is the given metric.

Let $f : \mathcal{X} \to \mathrm{Spec}(O_K)$ be an arithmetic surface as in the previous section and let $D : \mathrm{mSpec}(O_L) \to \mathcal{X}$ be a reduced divisor over $\mathcal{X}$.

Let $f_L : \mathcal{X}_L \to \mathrm{Spec}(O_L)$ be a desingularization of the arithmetic surface $\mathcal{X} \times_B \mathrm{Spec}(O_L)$. The base change of the morphism D give rise to a section $S_D : B_L \to \mathcal{X}_L$; moreover, if $p : \mathcal{X}_L \to \mathcal{X}$ is the natural projection, by construction we have that $p \circ S_D = D$.

3.1. Proposition. *Let $(\widehat{\mathcal{X}_L})_D$ be the completion of $\mathcal{X}_L$ around $S_D(B_L)$; then there is a natural isomorphism*

$$\Psi_D : (\widehat{\mathcal{X}_L})_D \to \mathrm{Spf}\big(\mathcal{O}\big[\!\big[\mathcal{O}(-S_D)|_{S_D}\big]\!\big]\big).$$

Proof. Since $\mathcal{X}_L$ is regular and S_D is a section, $S_D(B_L)$ is contained in the smooth open set of the structural morphism f_L. Consequently we can find an open neighborhood $\mathcal{U}$ of $S_D(B_L)$ in $\mathcal{X}_L$ and an *étale* map $g_D : \mathcal{U} \to \mathbb{V}(\mathcal{O}(-S_D))|_{S_D}$ sending $S_D(B_L)$ to the zero section. From this the proposition follows. □

Let $\mathcal{X}_i$ $(i = 1, 2)$ be the arithmetic surfaces fixed in the previous section. Let $D_1 : \mathrm{Spec}(O_{L_1}) \to \mathcal{X}_1$ and $D_2 : \mathrm{Spec}(O_{L_2}) \to \mathcal{X}_2$ be effective reduced divisors on $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively; let $L := L_1 \cdot L_2$ be the composite of L_1 and L_2 over K. As before they define two sections $S_i : \mathrm{Spec}(O_L) \to (\mathcal{X}_i)_{O_L}$ $(i = 1, 2)$. Let $\xi_{D_1,D_2} : B_L \to (\mathcal{X}_1 \times \mathcal{X}_2)_L$ be the point obtained from S_1 and S_2 and denote by $(\widehat{\mathcal{X}_1 \times \mathcal{X}_2})_{\xi_{D_1,D_2}}$ the completion of $(\mathcal{X}_1 \times \mathcal{X}_2)_L$ around ξ_{D_1,D_2}. As corollary of 3.1 we obtain

3.2. Corollary. *Let $(\widehat{\mathcal{X}_1 \times \mathcal{X}_2})_{\xi_{D_1,D_2}}$ the completion of $(\mathcal{X}_1 \times \mathcal{X}_2)_L$ around ξ_{D_1,D_2}. Then there is a natural isomorphism*

$$\Psi_{D_1,D_2} : (\widehat{\mathcal{X}_1 \times \mathcal{X}_2})_{\xi_{D_1,D_2}} \to \mathrm{Spf}\big(\mathcal{O}\big[\!\big[\mathcal{O}(-S_1)|_{S_1}; \mathcal{O}(-S_2)|_{S_2}\big]\!\big]\big).$$

Let M_L be the set of places of L; and $\sigma \in M_L$ be an infinite place. As explained before, the O_L-algebra $(\mathcal{O}[(\mathcal{O}(-S_1)|_{S_1}; \mathcal{O}(-S_2)|_{S_2}])_\sigma$ is naturally equipped with the structure of *hermitian* vector space because of the choice of the metrics as in Section 1.

If $p_i : (\mathcal{X}_1)_L \times (\mathcal{X}_2)_L \to (\mathcal{X}_i)_L$ is the natural projection, and N is a line bundle on $(\mathcal{X}_i)_L$, by abuse of notation, we will denote again by N the line bundle $p_i^*(N)$ on $(\mathcal{X}_1)_L \times (\mathcal{X}_2)_L$.

In this section we will construct sections of small norm of suitable line bundles with high order of vanishing along $\xi_{1,2}$. As usual the key lemma is the Siegel lemma. Before we give the statement (and the proof) of the Siegel lemma we need, we recall without proof all the tools we need; for the proofs we refer to [Bo, §4.1] and [Sz]:

(a) if E is an hermitian vector bundle over O_K, then we call the real number $\mu_n(E) := \frac{1}{[K;\mathbb{Q}]} \cdot \frac{\widehat{\deg}(E)}{rk(E)}$, *the slope* of E;
(b) within all the sub bundles of a given hermitian vector bundle E, there is one F having maximal slope; we call its slope *the maximal slope* of E and denote it by $\mu_{\max}(E)$; if $F = E$ we will say that E is *semistable*; by construction the sub bundle F is semistable;
(c) if E_1 and E_2 are two hermitian vector bundles, we have that $\mu_{\max}(E_1 \oplus E_2) = \max\{\mu_{\max}(E_1); \mu_{\max}(E_2)\}$;
(d) let $f : E \to F$ be an *injective* morphism between hermitian vector bundles; then $\frac{1}{[K:\mathbb{Q}]}\widehat{\deg}(E) \leqslant rk(E)(\mu_{\max}(F) + \log\|f\|)$;
(e) there is a constant $\chi(K)$ depending only on K (for the precise value we refer to [Sz]) such that, if E is an hermitian vector bundle on K with $\widehat{\deg}(E) > -rk(E)\chi(K)$, then there is a non-torsion element $v \in E$ such that, for every infinite place σ we have $\sup_{\sigma \in M_\infty}\{\log(\|v\|_\sigma)\} \leqslant 3\log(rk(E))$; we define $\|\cdot\|_{\sup}$ to be $\sup\{\|\cdot\|_\sigma\}_{\sigma\in M_\infty}$ (cf. [BGS, Theorem 5.2.4] and below it);
(f) let M_∞ be the set of infinite places of K and $\lambda := (\lambda_\sigma)_{\sigma\in M_\infty}$ be an element of $\mathbb{R}^{[K:\mathbb{Q}]}$ with $\lambda_\sigma = \lambda_{\bar\sigma}$; we denote by $\mathcal{O}(\lambda)$ the hermitian line bundle $(O_K, \|1\|_\sigma = \exp(-\lambda_\sigma))$. If E is an hermitian vector bundle over O_K then we denote by $E(\lambda)$ the hermitian vector bundle $E \otimes \mathcal{O}(\lambda)$.
(g) (Hilbert–Samuel formula) there is a constant C, depending on the choices made (but not on the d_i's), such that, if d_1 and d_2 are sufficiently big, the Hermitian O_K-module $H^0 = (\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} \otimes M_2^{d_2})$ is generated by elements of sup-norm, less or equal then $C^{d_1+d_2}$.

We will also need the following

3.3. Lemma. *Let*

$$0 \to E_1 \to E \to E_2 \to 0$$

be an exact sequence of hermitian vector bundles; then

$$\mu_{\max}(E) \leqslant \max\{\mu_{\max}(E_1), \mu_{\max}(E_2)\}.$$

The proof is straightforward and left to the reader.

Let $K \subseteq L$ be a finite extension and $\mathrm{Spec}(O_L) \to \mathrm{Spec}(O_K)$ the induced morphism. Let F be an hermitian vector bundle on $\mathrm{Spec}(O_L)$. Observe that the vector bundle $f_*(F)$ over $\mathrm{Spec}(O_K)$ is naturally equipped with the structure of hermitian vector bundle.

3.4. Lemma. *Suppose that F is equipped with a filtration $F = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_N = 0$ with F_i/F_{i+1} line bundles, equipped with the induced hermitian metric. Then*

$$\mu_{\max}(f_*(F)) \leqslant \frac{1}{[L:\mathbb{Q}]}\max\{\widehat{\deg}(F_i/F_{i+1})\}.$$

Proof. By devissage we are reduced to prove it when F is itself a line bundle. Let $Q \subseteq f_*(F)$ be the maximal semistable subbundle. We deduce a map $f^*(Q) \to F$ consequently an isometric inclusion of O_L in $f^*(Q^\vee) \otimes F$. Thus we get $\mu_n(f^*(Q^\vee) \otimes F) \geqslant 0$ *because $f^*(Q^\vee)$ is semistable*. So $[L:\mathbb{Q}]\mu_n(Q) \leqslant \deg(F)$. The conclusion follows. □

The Siegel lemma we need is the following:

3.5. Lemma *(Siegel lemma). Let V and W be hermitian vector bundles over $\mathcal{O}_K$. Let $\gamma : V \to W$ be a* non-injective *morphism. Let $m = rk(V)$ and $n := rk(\mathrm{Ker}(\gamma))$. Suppose that there is a constant $C > 1$ such that*:

(i) *V is generated by elements with* sup *norm at most C;*
(ii) *$\|\gamma\| \leqslant C$, then, there is a non-zero element $x \in \mathrm{Ker}(\gamma)$ with*

$$\sup_{\sigma\in M_\infty}\big\{\log\big(\|x\|_\sigma\big)\big\} \leqslant \frac{m}{n}\cdot\log\big(C^2\big) + \left(\frac{m}{n}-1\right)\mu_{\max}(W) - \frac{\chi(K)}{[K:\mathbb{Q}]} + 3\log(n).$$

Proof. Denote by U the hermitian vector bundle $\mathrm{Ker}(\gamma)$ with the induced metric. Observe that, by property (e) above, if $\widehat{\deg}(U(\lambda)) > -n\chi(K)$, then there is a non-torsion element $x \in U$ such that

$$\sup_{\sigma\in M_\infty}\big\{\log\big(\|x\|_\sigma\big)\big\} \leqslant \sup_{\sigma\in M_\infty}\{\lambda_\sigma\} + 3\log\big(rk(U)\big).$$

An easy computation gives $\widehat{\deg}(U(\lambda)) = \widehat{\deg}(U) + n\cdot\sum_\sigma \lambda_\sigma$. Let W' be the image of γ. Put on W' the metric induced by the surjection. Thus we have

$$\widehat{\deg}\big(U(\lambda)\big) = \widehat{\deg}(V) - \widehat{\deg}(W') + n\cdot\sum_\sigma \lambda_\sigma.$$

By property (d) we have $\frac{\widehat{\deg}(W')}{[K:\mathbb{Q}]} \leqslant (m-n)(\mu_{\max}(W) + \log(C))$ and by the very definition of Arakelov degree, $\widehat{\deg}(V) \geqslant -m[K:\mathbb{Q}]\log(C)$. Consequently

$$\begin{aligned}\widehat{\deg}\big(U(\lambda)\big) &= \widehat{\deg}(V) - \widehat{\deg}(W') + n\cdot\sum_{\sigma\in M_\infty}\lambda_\sigma\\ &\geqslant -2m[K:\mathbb{Q}]\log(C) - (m-n)[K:\mathbb{Q}]\mu_{\max}(W) + n\cdot\sum_{\sigma\in M_\infty}\lambda_\sigma;\end{aligned}$$

thus, take $\lambda_\sigma = \frac{m}{n}\cdot\log(C^2) + (\frac{m}{n}-1)\mu_{\max}(W) - \frac{\chi(K)}{[K:\mathbb{Q}]} + \epsilon$ and apply the observation above. The conclusion follows. $\square$

Let ϑ_1, ϑ_2 and δ be three positive rational numbers. For every couple of positive integers (d_1, d_2) we denote by $\mathcal{I}_{\underline{\vartheta},\delta,\underline{d}}$ the ideal sheaf of $(\mathcal{X}_1)_L \times (\mathcal{X}_2)_L$ defined by

$$\sum_{\substack{\frac{i}{d_1}\cdot\vartheta_1+\frac{j}{d_2}\cdot\vartheta_2\geqslant\delta\\ i\leqslant d_1,\ j\leqslant d_2}} \mathcal{O}(-iS_1)\otimes\mathcal{O}(-jS_2). \tag{3.6.1}$$

In the same way, we will denote by $I_{\underline{\vartheta},\delta,\underline{d}}$ the ideal of $\mathcal{O}[\mathcal{O}(-S_1)|_{S_1}, \mathcal{O}(-S_2)|_{S_2}]$ defined by a condition analogous to condition (3.6.1).

We denote by $\mathcal{A}_{\underline{\vartheta},\delta,\underline{d}}$ the subscheme of $(\mathcal{X}_1)_L \times (\mathcal{X}_2)_L$ defined by the ideal $\mathcal{I}_{\underline{\vartheta},\delta,\underline{d}}$ and by $W_{\underline{\vartheta},\delta,\underline{d}}$ the O_L algebra $\mathcal{O}[\mathcal{O}(-S_1)|_{S_1}, \mathcal{O}(-S_2)|_{S_2}]/I_{\underline{\vartheta},\delta,\underline{d}}$. Then:

(i) the isomorphism $\Psi_{D_1^{h_1},D_2^{h_2}}$ induces an isomorphism $\Psi_{1,2} : \mathcal{A}_{\underline{\vartheta},\delta,\underline{d}} \to \mathrm{Spec}(W_{\underline{\vartheta},\delta,\underline{d}})$;
(ii) the O_L module $W_{\underline{\vartheta},\delta,\underline{d}}$ has a natural structure of hermitian O_L-module. Moreover the O_L-module $W_{\underline{\vartheta},\delta,\underline{d}}$ has a filtration by O_L-submodules F^μ such that $F^\mu/F^{\mu+1} \simeq \mathcal{O}(-iS_1)|S_1 \otimes \mathcal{O}(-jS_2)|_{S_2}$ with $\frac{i}{d_1}\cdot\vartheta_1 + \frac{j}{d_2}\cdot\vartheta_2 \leqslant \delta$; this filtration is isometric.

3.6. Proposition. *Let $\epsilon > 0$ and $\delta > 2$ be two rational numbers; suppose that $2 \cdot \vartheta_1 \cdot \vartheta_2 > \delta^2[L:K] + \epsilon$, then there exists a constant A depending only on $\mathcal{X}_i$, M_i, $[L:K]$, ϑ_i and ϵ such that the following holds.*

For every couple of irreducible divisor $D_1 \hookrightarrow \mathcal{X}_1$, and $D_2 \hookrightarrow \mathcal{X}_2$ as above and every couple of sufficiently big integers (d_1, d_2), there is a non-zero section $f \in H^0(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} \otimes M_2^{d_2})$ vanishing along $\mathcal{A}_{\underline{\vartheta},\delta,\underline{d}}$ and such that, for every infinite place $\sigma \in M_K$ we have

$$\log\bigl(\|f\|_\sigma\bigr) \leqslant \frac{A}{\epsilon} \cdot T(D_1) \cdot T(D_2) \cdot (d_1 + d_2);$$

where the $T(D_i)$ are defined as in Section 2.

Proof. Let $\gamma : \mathrm{Spec}(O_L) \to \mathrm{Spec}(O_K)$ the morphism induced by the inclusion $K \subseteq L$. It induces a morphism of hermitian modules

$$\gamma_{d_1,d_2} : H^0\bigl(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} \times M_2^{d_2}\bigr) \to \gamma_*\bigl(W_{\underline{\vartheta},\delta,\underline{d}} \otimes \bigl(M_1^{d_1}\bigr)\big|_{S_1} \otimes \bigl(M_2^{d_2}\bigr)\big|_{S_2}\bigr).$$

Let $K(d_1, d_2)$ be the kernel of γ_{d_1,d_2}. We have to prove that there exists an element in $K(d_1, d_2)$ having bounded norm.

In the sequel of this proof, "absolute constant" will be equivalent to say "a constant which depends only on the $\mathcal{X}_i$, on the hermitian line bundles $\overline{M}_i$ and on the metrics on the diagonals; but *independent on the D_i's* and on the d_i's."

By Lemmas 3.3, 3.4 and (ii) above we can find an absolute constant A such that $\mu_{\max}(\gamma_*(W_{\underline{\vartheta},\delta,\underline{d}} \otimes (M_1^{d_1})|_{S_1} \otimes (M_2^{d_2})|_{S_2})) \leqslant A \cdot T(D_1) \cdot T(D_2) \cdot (d_1 + d_2)$.

By (g) above, as soon as d_1 and d_2 are sufficiently big, we can find an absolute constant A for which $H^0(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} \otimes M_2^{d_2})$ is generated by elements with norm bounded by $A^{d_1+d_2}$.

Now we come to the main part of the proof: we can find an absolute constant C for which the O_K-module $H^0(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} \otimes M_2^{d_2})$ has rank which is bounded below by $C \cdot d_1 \cdot d_2$. The rank of the O_L-module $W_{\underline{\vartheta},\delta,\underline{d}} \otimes (M_1^{d_1})|_{S_1} \otimes (M_2^{d_2})|_{S_2}$ can be bounded from above as follows: the number of the terms of the filtration described in (ii) is the number of couples of positive integers (i, j) with $i \leqslant d_1$, $j \leqslant d_2$ and $\frac{i}{d_1} \cdot \vartheta_1 + \frac{j}{d_2} \cdot \vartheta_2 \leqslant \delta$; as soon as d_1 and d_2 are sufficiently big, this number is bounded above by $d_1 \cdot d_2$ multiplied by the area of the triangle with vertices $(0,0)$, $(\frac{\delta}{\vartheta_1}, 0)$, and $(0, \frac{\delta}{\vartheta_2})$ plus a very small error term, consequently

$$rk_{O_K}\bigl(W_{\underline{\vartheta},\delta,\underline{d}} \otimes \bigl(M_1^{d_1}\bigr)\big|_{S_1} \otimes \bigl(M_2^{d_2}\bigr)\big|_{S_2}\bigr) \leqslant d_1 \cdot d_2 \cdot \frac{\delta^2}{2\vartheta_1 \cdot \vartheta_2}[L:K] + \epsilon'.$$

Consequently there is an absolute constant A such that

$$\frac{rk_{O_K}(h^0(\mathcal{X}_1 \otimes \mathcal{X}_2, M_1^{d_1} \times M_2^{d_2}))}{rk_{O_K}(K(d_1, d_2))} \leqslant \frac{A}{\epsilon}.$$

For every infinite place σ of K, we cover the Riemann surface $\mathcal{X}_{i,\sigma}$ with a finite number of disks over which the line bundle M_i trivializes; inside each disk we take a disk with same center and radius one half of the radius of it; we may suppose that also these smaller disks cover the Riemann surface (we suppose that this covering is fixed once for all, in particularly independently of the D_i's). From Lemma 3.7 below we deduce that we can find a constant A, independent on the D_i's, such that for every infinite place σ we have $\|\gamma_{d_1,d_2}\|_\sigma \leqslant A^{d_1+d_2}$. We may suppose that the d_i's are so big that $\log(d_i) \leqslant \epsilon d_i$. We apply now 3.5 to this situation and conclude the proof of the proposition. □

3.7. Lemma. *Let Δ_R be the disk of radius R. Let $f(x, y)$ be a holomorphic function on $\Delta_R \times \Delta_R$ and $(z_1, z_2) \in \Delta_{R/2} \times \Delta_{R/2}$ then for every (i, j),*

$$\left| \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(z_1, z_2) \right| \leqslant \frac{2^{i+j} i! j!}{R^{i+j}} \cdot \max_{\|x\|=\|y\|=R} \{ |f(x, y)| \}.$$

The proof of the lemma is a straightforward application of the maximum modulus principle and the Cauchy inequality.

4. Index theorem

In this section we prove that, under suitable hypotheses, the section of 3.5 has a small order of vanishing along a point verifying the inequality of the main theorem. We will prove an analogue of the "Roth index theorem" in this context.

4.1. Remark. In a first version of the paper we deduced the index theorem from a generalization of the Vojta version of Dyson lemma for curves [Vo1]; but, due to the "admissibility hypothesis" in this kind of theorems, this could be applied only in the case when *both* the D_i's have the same degree.

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be the arithmetic surfaces. Let M be a line bundle over $(\mathcal{X}_1 \times \mathcal{X}_2)_K$ and $f \in H^0((\mathcal{X}_1 \times \mathcal{X}_2)_K, M)$. We fix two positive rational numbers $\vartheta_i \geqslant 1$.

Let d_1 and d_2 be two positive integers such that $d_i/\vartheta_i \in \mathbb{Z}$.

Let $P := (P_1, P_2) \in (\mathcal{X}_1 \times \mathcal{X}_2)_K(K)$ be a point and z_i be local coordinate around P_i in $X_i := (\mathcal{X}_i)_K$ $(i = 1, 2)$. Let e be a local generator of M around P; consequently, near P, we can write $f = g \cdot e$ where g is a regular function around P. We will say that f has index at least δ in P with respect to d_1 and d_2 and we will write $\mathrm{ind}_P(f, d_1, d_2) \geqslant \delta$ if, near P, we write $g = \sum_{i,j} a_{i,j} z_1^i \cdot z_2^j$ and $a_{i,j} = 0$ whenever

$$\frac{i}{d_1} \cdot \vartheta_1 + \frac{j}{d_2} \cdot \vartheta_2 < \delta.$$

The definition of the index is independent on the choices.

The condition $\mathrm{ind}_P(f, d_1, d_2) \geqslant \delta$ defines a closed subscheme of $(\mathcal{X}_1 \times \mathcal{X}_2)_K$ which will be denoted by $Z_\delta(f)$ (in the notation, the dependence on the d_i's will be clear from the context).

Let M_i be the line bundles of generic degree one on $\mathcal{X}_i$ $(i = 1, 2)$ fixed in the previous section. As in the previous section we will denote by M_i the line bundle $pr_i^*(M_i)$ on $X_1 \times X_2$ $(pr_i : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_i$ being the natural projection).

The main theorem of this section is

4.2. Theorem. *Let C and ϵ be positive real numbers. Then we can find constants $B_j = B_j(C, \epsilon)$ depending only on C, the ϑ_1, and ϵ (and on the other choices made until now), but independent on the d_i's, having the following property*:

Suppose that:

(a) *$f \in H^0(\mathcal{X}_1 \times \mathcal{X}_2; M_1^{d_1} \otimes M_2^{d_2})$ is a global section with $\sup_{\sigma \in M_\infty} \{\|f\|_\sigma\} \leqslant C^{(d_1+d_2)}$*;
(b) *the d_i's are sufficiently big and divisible and $d_1/d_2 \geqslant B_1$*;
(c) *$P := (P_1, P_2) \in \mathcal{X}_1 \times \mathcal{X}_2(K)$ is a rational point such that*

$$B_2 \leqslant h_{M_1}(P_1) \quad \text{and} \quad \frac{h_{M_2}(P_2)}{h_{M_1}(P_1)} \geqslant \frac{d_1}{d_2},$$

then

$$\mathrm{ind}_P(f, d_1, d_2) \leqslant \epsilon.$$

4.3. Remark. The proof of the statement above is directly inspired by the Faltings product theorem [Fa2] and can be deduced from it; we propose here a self contained proof (which is simpler then the proof of the product theorem in this situation).

One can develop a height for subvarieties of a fixed variety (cf. [BGS]); this theory extends the height theory for points. We will not recall here the definitions but we will recall the properties of the heights that we need. Indeed, the only things we need of the theories are the properties quoted below (consequently a reader who do not know the theory can simply admit them).

We will use the following standard facts from the height theory of subvarieties, one can find the proofs on [Fa2] or on [Ev]; if Z is a closed subscheme of $\mathcal{X}_1 \times \mathcal{X}_2$ and M is an hermitian line bundle, then we denote by $h_M(Z)$ the height of Z with respect to M as defined in [BGS]; by definition the height of a closed subscheme is a real number. By linearity, the height function is also defined on cycles:

(a) Suppose that Z_i are closed irreducible reduced subschemes of $\mathcal{X}_i$ of relative dimension δ_i (over $\mathbb{Z}$) then

$$h_{M_1^{d_1} \otimes M_2^{d_2}}(Z_1 \times Z_2) = (\delta_1 + \delta_2 + 1)! \cdot d_1^{\delta_1} \cdot d_2^{\delta_2} \cdot \left(\frac{d_1 \cdot h_{M_1}(Z_1)}{(\delta_1 + 1)!} + \frac{d_2 \cdot h_{M_2}(Z_2)}{(\delta_2 + 1)!} \right);$$

this is proved in [Ev, Lemma 8].

(b) Suppose that $\mathcal{X}_i = \mathbb{P}^1$ and $M_i = \mathcal{O}(1)$ and $C > 1$ is a real constant. Then there is a constant S, depending only on $\mathcal{X}_i$ and the chosen metrics (but independent on the d_i's and on C), such that the following holds: let $f_1, \ldots, f_r \in H^0(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{O}(d_1) \otimes \mathcal{O}(d_2))$ be integral global sections such that $\sup_{\sigma \in M_\infty}\{\|f_i\|_\sigma\} \leqslant C^{(d_1+d_2)}$; let Y be the subscheme of $\mathcal{X}_1 \times \mathcal{X}_2$ defined as the zero set of $\{f_1, \ldots, f_r\}$; let X be an irreducible component of Y with multiplicity m_X then

$$m_X \cdot h_{\mathcal{O}(d_1) \otimes \mathcal{O}(d_2)}(X) \leqslant S \cdot \log(C) \cdot d_1 \cdot d_2 \cdot (d_1 + d_2);$$

this is proved in [Fa2, Proposition 2.17] or [Ev, Lemma 9].

(c) If $f \in H^0(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} \otimes M_2^{d_2})$ then

$$h_{M_1^{d_1} \otimes M_2^{d_2}}\big(\mathrm{div}(f)\big) = h_{M_1^{d_1} \otimes M_2^{d_2}}(\mathcal{X}_1 \times \mathcal{X}_2) + \sum_{\sigma \in M_\infty} \int_{(\mathcal{X}_1 \times \mathcal{X}_2)_\sigma} \log \|f\|_\sigma \big(c_1\big(M_1^{d_1} \otimes M_2^{d_2}\big)_\sigma\big)^2;$$

this is a direct consequence of the definition of height (cf. [BGS]); consequently (using point (a)), we can find a positive constant S, depending only on the $\mathcal{X}_i$'s the M_i's and the chosen metrics, for which the following holds: let $C > 1$ be a constant; if $f \in H^0(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} \otimes M_2^{d_2})$ is such that $\sup_{\sigma \in M_\infty}\{\|f\|_\sigma\} \leqslant C^{(d_1+d_2)}$ then

$$h_{M_1^{d_1} \otimes M_2^{d_2}}\big(\mathrm{div}(f)\big) \leqslant S \cdot \log(C) \cdot d_1 \cdot d_2 \cdot (d_1 + d_2).$$

Proof of 4.2. Let f be the given section and Z be a geometrically irreducible reduced component of $Z_\epsilon(f)$. Extending K if necessary, we may suppose that Z is defined over K. It suffices to prove that, under the hypotheses of the theorem (with explicit and suitable B_i's) the point P do not belong to Z. There are two cases, depending on the dimension of Z.

Case 1: *Dimension of Z equal to one.* Let $Y := \mathrm{div}(f)$; since Z is a divisor contained in Y we have

$$Y_K = m_Z \cdot Z + D;$$

where D is an effective divisor on $(\mathcal{X}_1 \times \mathcal{X}_2)_K$. We claim that, if $d_1/d_2 \geqslant \vartheta_1/(\epsilon \cdot \vartheta_2)$ then either there is a point $A \in \mathcal{X}_2(K)$ such that $Z = (\mathcal{X}_1)_K \times \{A\}$, or there is a point $B \in \mathcal{X}_1(K)$ such that $Z = \{B\} \times (\mathcal{X}_2)_K$.

4.4. Lemma. *Suppose that Z is* not *as claimed, then $m_Z \geqslant \epsilon \cdot \frac{d_1}{\vartheta_1}$.*

Let us show how the lemma implies the claim: Suppose that Z is not as claimed, then $(Z; M_1) > 0$; consequently, denoting by $(\cdot;\cdot)$ the intersection pairing on the surface $(\mathcal{X}_1 \times \mathcal{X}_2)_K$,

$$d_2 = (Y; M_1) \geqslant \epsilon \frac{d_1}{\vartheta 1}(Z; M_1) > \epsilon \cdot \frac{d_1}{\vartheta_1},$$

so $d_1/d_2 \leqslant \vartheta_1/\epsilon$; thus, taking $B_1 > \frac{\vartheta_1}{\epsilon}$, we find a contradiction.

Proof of the lemma. Let η be a generic point of Z not contained in D; we may suppose that the restriction of both projections are étale in a neighborhood of η. Let z_1 and z_2 be local coordinates about the projections of η. In a formal neighborhood of η, the divisor Z is defined by an irreducible element $h \in K[[z_1, z_2]]$ and Y is defined by the ideal (h^{m_Z}); because of our choice of η, we have $h(z_1, z_2) = a_{10}z_1 + a_{01}z_2 + O((z_1 + z_2)^2)$ with $a_{01} \cdot a_{10} \neq 0$, moreover, by definition of $Z_\epsilon(f)$,

$$\bigl(h(z_1, z_2)\bigr)^{m_Z} = \sum_{i,j} b_{ij} \cdot z_1^i \cdot z_2^j$$

with $b_{ij} = 0$ whenever $\frac{i}{d_1} \cdot \vartheta_1 + \frac{j}{d_2} \cdot \vartheta_2 \leqslant \epsilon$. Observe that $b_{m_Z,0} = a_{01}^{m_Z} \neq 0$ thus

$$m_Z \geqslant \epsilon \cdot \frac{d_1}{\vartheta_1}. \qquad \square$$

We thank the referee whose suggestions helped to drastically simplify the proof of the lemma above.

We now come to the arithmetic part of the proof, in this case: Z is either $(\mathcal{X}_1)_K \times \{A\}$ or $\{B\} \times (\mathcal{X}_2)_K$ for suitable A and B; remark that in the first case $A = P_2$ and in the second case $B = P_1$. It is easy to see that m_Z is exactly $\epsilon \cdot \frac{d_2}{\vartheta_2}$ in the first case and exactly $\epsilon \cdot \frac{d_1}{\vartheta_1}$ in the second: indeed it suffices to compute m_Z on a smooth point of the support of Z and Y. In the first case, by applying properties (a) and (c) above, the fact that the height is additive on cycles and the hypotheses, we can find an explicit constant R_1 depending only on C such that:

$$\begin{aligned} m_Z \cdot h_{M_1^{d_1} \otimes M_2^{d_2}}(Z) &= m_Z \cdot d_1\bigl(d_1 \cdot h_{M_1}(\mathcal{X}_1) + 2d_2 \cdot h_{M_2}(A)\bigr) \\ &\leqslant h_{M_1^{d_1} \otimes M_2^{d_2}}\bigl(\mathrm{div}(f)\bigr) \leqslant R_1 \cdot d_1 \cdot d_2(d_1 + d_2); \end{aligned}$$

consequently, since $m_Z = \epsilon \cdot \frac{d_2}{\vartheta_2}$,

$$\frac{\epsilon \cdot d_1 \cdot d_2}{\vartheta_2} \cdot \bigl(d_1 \cdot h_{M_1}(\mathcal{X}_1) + 2d_2 \cdot h_{M_2}(A)\bigr) \leqslant R_1 \cdot d_1 \cdot d_2(d_1 + d_2),$$

thus

$$d_2 \cdot h_{M_2}(A) < \frac{R_2 \cdot \vartheta_2}{\epsilon}(d_1 + d_2).$$

Similarly, in the second case, we obtain

$$\frac{\epsilon \cdot d_1 \cdot d_2}{\vartheta_1} \cdot \big(2d_1 \cdot h_{M_1}(B) + d_2 \cdot h_{M_2}(\mathcal{X}_2)\big) \leqslant R_1 \cdot d_1 \cdot d_2 \cdot (d_1 + d_2),$$

thus

$$h_{M_1}(B) \leqslant \frac{R_2}{\epsilon} \cdot \left(1 + \frac{d_2}{d_1}\right).$$

This implies that, if $d_1/d_2 \geqslant 1$, the point (P_1, P_2) cannot be on Z as soon as $\frac{d_2}{d_1} \cdot h_{M_2}(P_2) \geqslant h_{M_1}(P_1) \geqslant \frac{2R_2}{\epsilon} \cdot \max\{\vartheta_1, \vartheta_2\}$.

Case 2: *Dimension of Z equal to zero.* Denote by $(P, Q) \in (\mathcal{X}_1 \times \mathcal{X}_2)_K(K)$ the support of Z. In this case we need to project on $\mathbb{P}^1 \times \mathbb{P}^1$. We fix once for all a finite set of coverings $\gamma_{i,j} : (\mathcal{X}_i)_K \to \mathbb{P}^1$ with the following property: if $U_{i,j} \subseteq (\mathcal{X}_i)_K$ is the open set over which $\gamma_{i,j}$ is étale, then $\bigcup_j U_{i,j} = (\mathcal{X}_i)_K$ and $\gamma_{i,j}^*(\mathcal{O}(1)) \simeq (M_i)_K^{t_i}$ for suitable t_i (we fix such isomorphisms). We also suppose that each $\gamma_{i,j}$ extends to a generically finite morphism $\gamma_{i,j} : \mathcal{X}_i \to \mathbb{P}^1_{O_K}$ (this can be obtained after a suitable blow-up of $\mathcal{X}_i$). We equip the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$ with the Fubini–Study metric $\|\cdot\|_{FS}$. Fix a constant A such that

$$A^{-1}\gamma_{i,j}^*\big(\|\cdot\|_{FS}\big) \leqslant \|\cdot\|_{M_i}^{t_i} \leqslant A\gamma_{i,j}^*\big(\|\cdot\|_{FS}\big).$$

We may suppose that $(P, Q) \in (\mathcal{X}_1 \times \mathcal{X}_2)(K)$ is contained in $U_{1,1} \times U_{2,1}$. Denote by Γ the morphism $\gamma_{1,1} \times \gamma_{1,2} : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{P}^1 \times \mathbb{P}^1$. Put $d_i = t_i \cdot a_i$; then $\Gamma^*(\mathcal{O}(a_1, a_2)) = M_1^{d_1} \otimes M_2^{d_2}$ and $g := \Gamma_*(f) \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d_1 \cdot t_2, d_2 \cdot t_1))$. It is easy to verify that there exists an absolute constant A_1 such that

$$\|g\|_{FS} \leqslant A_1^{(d_1+d_2)}\|f\|$$

and that $(P', Q') := \Gamma(P, Q)$ is contained in $Z_\epsilon(g)$. Consequently it suffices to prove the theorem when $\mathcal{X}_1 = \mathcal{X}_2 = \mathbb{P}^1$, $M_1 = M_2 = \mathcal{O}(1)$ and $\mathcal{O}(1)$ is equipped with the Fubini–Study metric.

We first look to the irreducible components Z' of $Z_{\epsilon/2}$ containing (P', Q'). If there is such a Z' of dimension one, then we are reduced to the previous case and we are done. We may then suppose that the support of Z' is (P', Q') too. Let I_ϵ and $I_{\epsilon/2}$ be the ideal of Z and Z' in the completion $K[[z_1, z_2]]$ of the local ring of $\mathbb{P}^1 \times \mathbb{P}^1$ in (P', Q'); let $h = \alpha \cdot z_1^{r_1} z_2^{r_2} + \cdots$ be an element of $I_{\epsilon/2}$ then

$$\frac{\partial^{i_1+i_2}}{\partial z_1^{i_1} \cdot \partial z_2^{i_2}} h = \alpha_1 z_1^{(r_1-i_1)'} z_2^{(r_2-i_2)'} + \cdots$$

(where $(a)' := \sup\{a, 0\}$) for a suitable α_1; and α_1 is zero only if α is zero or $\alpha \neq 0$ and one of the $(r_j - i_j)'$ is zero. If $\frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 < \frac{\epsilon}{2}$ and $h \in I_{\epsilon/2}$ then $\frac{\partial^{i_1+i_2}}{\partial z_1^{i_1} \cdot \partial z_2^{i_2}} h \in I_\epsilon \subseteq (z_1, z_2)$. This implies that h, and consequently $I_{\epsilon/2}$, is contained in the ideal $(z_1^{\epsilon d_1/(4\cdot\vartheta_1)}, z_2^{\epsilon d_2/(4\vartheta_2)})$. Thus, the multiplicity of Z' in $Z_{\epsilon/2}(g)$ is at least $\frac{1}{\vartheta_1 \cdot \vartheta_2} \cdot (\frac{\epsilon}{4})^2 \cdot d_1 d_2$.

Every differential operator $\frac{\partial^{i_1+i_2}}{\partial z_1^{i_1} \partial z_2^{i_2}}$ with $\frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \leqslant \frac{\epsilon}{2}$ can be seen as a linear endomorphism $D^{(i_1,i_2)}$ of $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d_1, d_2))$. For every infinite place $\sigma \in M_\infty$ the norm of the operator $D^{(i_1,i_2)}$

(with i_1 and i_2 bounded as above) is bounded from above by $2^{\max\{\vartheta_1,\vartheta_2\}\cdot(d_1+d_2)}$. We apply property (b) above and the hypotheses and we find a constant R', depending only on C,

$$m_{Z'}\cdot\big(d_1\cdot h_{\mathcal{O}(1)}(P')+d_2\cdot h_{\mathcal{O}(1)}(Q')\big)\leqslant R'\cdot d_1\cdot d_2\cdot(d_1+d_2);$$

consequently, since $m_{Z'}\geqslant\frac{1}{\vartheta_1\cdot\vartheta_2}\cdot(\frac{\epsilon}{4})^2\cdot d_1\cdot d_2$, we obtain

$$d_1\cdot h_{\mathcal{O}(1)}(P')+d_2\cdot h_{\mathcal{O}(1)}(Q')\leqslant\vartheta_1\cdot\vartheta_2\cdot\left(\frac{4}{\epsilon}\right)^2\cdot R'(d_1+d_2).$$

If we suppose that $\vartheta_1\cdot\vartheta_2\cdot(\frac{4}{\epsilon})^2\cdot R'\leqslant h_{\mathcal{O}(1)}(P_1)\leqslant h_{\mathcal{O}(1)}(P_2)$ the point P cannot belong to Z' and this concludes the proof of the lemma. $\square$

4.5. Remark. We observe that from the proof one deduce that the constants B_i's of Theorem 4.2 may be chosen of the form $B_i=S_i\log(C)$, where the S_i depend only on the $\mathcal{X}_j$ the M_i and the chosen metrics (but independent on C).

5. Generalized Cauchy inequalities

Fix $\vartheta_i\in\mathbb{Q}_{\geqslant 1}$ and the divisors $D_i:\mathrm{Spec}(O_L)\to\mathcal{X}_i$ as in Section 3. For every rational positive δ and couple of positive integers (d_1,d_2), let $\mathcal{I}_{\underline{\vartheta},\delta,\underline{d}}$ be the ideal sheaf of $\mathcal{X}_1\times\mathcal{X}_2$ defined in Section 3. Let $p:\tilde{\mathcal{X}}_\delta\to\mathcal{X}_1\times\mathcal{X}_2$ be the blow-up along $\mathcal{I}_{\underline{\vartheta},\delta,\underline{d}}$ and let E_δ be the corresponding exceptional divisor on it. We can find a very small positive constant α such that, if the d_i are sufficiently big, there is a surjection

$$\beta_\delta:\bigoplus_{\delta\leqslant\frac{i_1}{d_1}\cdot\vartheta_1+\frac{i_2}{d_2}\cdot\vartheta_2\leqslant\delta+\alpha}\mathcal{O}(-i_1\cdot D_1)\otimes\mathcal{O}(-i_2\cdot D_2)\twoheadrightarrow\mathcal{I}_{\underline{\vartheta},\delta,\underline{d}}.$$

Observe that α is independent on the d_i's, provided that they are sufficiently big.

To simplify notations we will denote by H the set

$$\left\{(i_1,i_2)\in\mathbb{Z}\times\mathbb{Z}\;\middle|\;\delta\leqslant\frac{i_1}{d_1}\cdot\vartheta_1+\frac{i_2}{d_2}\cdot\vartheta_2\leqslant\delta+\alpha\right\}.$$

If M is an hermitian line bundle on $\mathcal{X}_1\times\mathcal{X}_2$, by abuse of notation, we will denote again by M the pull back of M to $\tilde{\mathcal{X}}_\delta$.

The surjection β_δ above induces a surjection

$$\beta_\delta:\bigoplus_{(i_1,i_2)\in H}\mathcal{O}(-i_1\cdot D_1)\otimes\mathcal{O}(-i_2\cdot D_2)\twoheadrightarrow\mathcal{O}_{\tilde{X}_\sigma}(-E_\delta);$$

consequently the line bundle $\mathcal{O}_{\tilde{X}_\sigma}(E_\delta)$ is naturally equipped with the structure of *hermitian line bundle*.

If $P_i\in\mathcal{X}_i(K)$ are K-rational points of $\mathcal{X}_i$, they extend to sections $P_i:B:=\mathrm{Spec}(O_K)\to\mathcal{X}_i$. We will denote by $P:B\to\mathcal{X}_1\times\mathcal{X}_2$ the section $P_1\times P_2$ and by $\tilde{P}:B\to\tilde{\mathcal{X}}_\delta$ the strict transform of P.

The theorem we want to prove in this section is the following:

5.1. Theorem. *Let M be an hermitian line bundle on $\mathcal{X}_1\times\mathcal{X}_2$ and A and ϵ be positive constants. There is a constant C depending only on A, on the models, the metrics, the ϑ_i's, etc., but independent on the d_i's for which the following holds.*

Let $f \in H^0(\mathcal{X}_1 \times \mathcal{X}_2, M \otimes \mathcal{I}_{\underline{\vartheta},\delta,\underline{d}})$ be a global section such that $\sup_{\sigma\in M_\infty}\{\|f\|_\sigma\} \leqslant A$. Let $P := P_1 \times P_2 : \mathrm{Spec}(O_K) \to \mathcal{X}_1 \times \mathcal{X}_2$ be a rational point such that $\mathrm{ind}_P(f, d_1, d_2) \leqslant \epsilon$; then there exists $\epsilon' \leqslant \epsilon$, two positive integers i_1 and i_2 such that $\frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \leqslant \epsilon$ and a non-zero *global section $\tilde{f} \in H^0(\tilde{P}, M \otimes \omega^{i_1}_{\mathcal{X}_1/B} \otimes \omega^{i_2}_{\mathcal{X}_2/B} \otimes \mathcal{O}(-E_{\delta-\epsilon'}))$ such that*

$$\sup_{\sigma\in M_\infty}\{\|\tilde{f}\|_\sigma\} \leqslant A \cdot C^{(d_1+d_2)}.$$

Before we start the proof of the theorem, we need to introduce some notations and some tools.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two line bundles on $\mathrm{Spec}(O_K)$. For every couple of positive integers (i_1, i_2) we define the differential operator

$$D^{(i_1,i_2)} : \mathcal{O}[\![\mathcal{L}_1, \mathcal{L}_2]\!] \to \mathcal{O}[\![\mathcal{L}_1, \mathcal{L}_2]\!] \otimes \mathcal{L}_1^{i_1} \otimes \mathcal{L}_2^{i_2}$$

in the following way: let e_1 (resp. e_2) be a local generator of $\mathcal{L}_1$ (resp. of $\mathcal{L}_2$) then we define

$$D^{(i_1,i_2)}\big(e_1^a \otimes e_2^b\big) := \begin{cases} \binom{a}{i_1} \cdot \binom{b}{i_2} \cdot e_1^{a-i_1} \otimes e_2^{b-i_2} \otimes (e_1^{i_1} \otimes e_2^{i_2}) & \text{if } a \geqslant i_1 \text{ and } b \geqslant i_2, \\ 0 & \text{otherwise;} \end{cases}$$

and extend it linearly to $\mathcal{O}[\![\mathcal{L}_1, \mathcal{L}_2]\!]$; one easily verify that this definition do not depends on the choice of the local generators. The module $\mathcal{O}[\![\mathcal{L}_1, \mathcal{L}_2]\!] \otimes \mathcal{L}_1^{i_1} \otimes \mathcal{L}_2^{i_2}$ has a natural structure of $\mathcal{O}[\![\mathcal{L}_1, \mathcal{L}_2]\!]$-module (multiplication on the right). One can easily verify that $D^{(i_1,i_2)}$ is a differential operator: it is O_K-linear (by definition) and it satisfy the (iterated) Leibnitz-rule; for instance $D^{n,0}(f \cdot g) = \sum \binom{n}{i} \cdot D^{(i,0)}(f) \cdot D^{(n-i,0)}(g)$ ($D^{(i,0)}(f) \in \mathcal{O}[\![\mathcal{L}_1, \mathcal{L}_2]\!] \otimes \mathcal{L}_1^i$ and $D^{(n-i,0)}(g) \in \mathcal{O}[\![\mathcal{L}_1, \mathcal{L}_2]\!] \otimes \mathcal{L}_1^{n-i}$), consequently $D^{(i,0)}(f) \cdot D^{(n-i,0)}(g) \in \mathcal{O}[\![\mathcal{L}_1, \mathcal{L}_2]\!] \otimes \mathcal{L}_1^n$.

If $\sigma \in M_\infty$ is an infinite place, then $\mathcal{O}[\![\mathcal{L}_1, \mathcal{L}_2]\!]_\sigma$ is (non-canonically) isomorphic to the ring of formal power series in two variables and the operators $D^{(a,b)}$ are the usual iterated derivatives.

Although it is not necessary, we will tacitly authorize ourself to pass to the Hilbert class field extension: consequently we will suppose that every line bundle on B is trivial; this is not necessary, but highly simplify the notations.

Denote by $(\widehat{\mathcal{X}_1 \times \mathcal{X}_2})_P$ the formal completion of $\mathcal{X}_1 \times \mathcal{X}_2$ around P. By 3.2, we find a canonical isomorphism

$$\Psi_P : (\widehat{\mathcal{X}_1 \times \mathcal{X}_2})_P \xrightarrow{\sim} \mathrm{Spf}\big(\mathcal{O}\big[\!\big[\mathcal{O}(-P_1)|_{P_1}, \mathcal{O}(-P_2)|_{P_2}\big]\!\big]\big).$$

We will denote by $I_P \subset \mathcal{O}[\![\mathcal{O}(-P_1)|_{P_1}, \mathcal{O}(-P_2)|_{P_2}]\!]$ the ideal corresponding to the ideal of definition of $(\widehat{\mathcal{X}_1 \times \mathcal{X}_2})_P$ defining the point section P (with the reduced structure).

Proof of Theorem 5.1. Let $p_i : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_i$ be the projection. Denote by $\mathcal{I}_{D_i}$ the restriction of the ideal sheaf $p_i^*(\mathcal{O}(-D_i))$ to $(\widehat{\mathcal{X}_1 \times \mathcal{X}_2})_P$. The image of $\mathcal{I}_{D_i}$ by Ψ is a principal ideal of $\mathcal{O}[\![\mathcal{O}(-P_1)|_{P_1}, \mathcal{O}(-P_2)|_{P_2}]\!]$ generated by an element G_i. If δ is a positive rational number, we denote then by $I_{\delta,\underline{d}} \subset \mathcal{O}[\![\mathcal{O}(-P_1)|_{P_1}, \mathcal{O}(-P_2)|_{P_2}]\!]$ the ideal generated by the elements $G_1^i \cdot G_2^i$ with $\frac{i}{d_1} \cdot \vartheta_1 + \frac{j}{d_2} \cdot \vartheta_2 \geqslant \delta$. The ideal $I_{\delta,\underline{d}}$ is the image, via Ψ, of the restriction to $(\widehat{\mathcal{X}_1 \times \mathcal{X}_2})_P$ of the ideal sheaf $\mathcal{I}_{\underline{\vartheta},\delta,\underline{d}}$. Consequently, a global section $f \in H^0(\mathcal{X}_1 \times \mathcal{X}_2; M \otimes \mathcal{I}_{\underline{\vartheta},\delta,\underline{d}})$ restricted to $(\widehat{\mathcal{X}_1 \times \mathcal{X}_2})_P$ will determine an element

$$F = \sum_{\frac{i}{d_1}\cdot\vartheta_1 + \frac{j}{d_2}\cdot\vartheta_2 \geqslant \delta} a_{ij} \cdot G_1^i \cdot G_2^j.$$

If (i_1, i_2) is a couple of indices such that $\frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \leqslant \epsilon$ then a direct computation using the iterated Leibnitz rule gives $D^{(i_1,i_2)}(F) \in I_{\delta-\epsilon,\underline{d}} \otimes M|_P \otimes \mathcal{O}(-i_1 P_1)|_{P_1} \otimes \mathcal{O}(-i_2 P_2)|_{P_2}$.

Since the index $\mathrm{ind}_P(f, d_1, d_2)$ of f at P is less or equal then ϵ, then we can find a couple of positive integers (i_1, i_2) such that $\frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \leqslant \epsilon$ and such that the class $\tilde{f}$ of $D^{(i_1,i_2)}(f)$ in $(\mathcal{O}[\![\mathcal{O}(-P_1)|_{P_1}, \mathcal{O}(-P_2)|_{P_2}]\!] \otimes M|_P \otimes \mathcal{O}(-i_1 P_1)|_{P_1} \otimes \mathcal{O}(-i_2 P_2)|_{P_2})/I_P \simeq H^0(P, M \otimes \mathcal{O}(-i_1 P_1)|_{P_1} \otimes \mathcal{O}(-i_2 P_2)|_{P_2})$ is *non-zero*. Thus, using adjunction formula, we find a *non-zero* section in $\tilde{f} \in H^0(P, M \otimes \omega^{i_1}_{\mathcal{X}_1/B}|_{P_1} \otimes \omega^{i_2}_{\mathcal{X}_2/B}|_{P_2} \otimes \mathcal{I}_{\underline{\vartheta},\delta-\epsilon,\underline{d}})$.

Let $\sigma \in M_\infty$ be an infinite place. We fix once for all a covering of $(\mathcal{X}_i)_\sigma$ by open sets U_{ij} analytically equivalent to a disk (with coordinate z) for which the following holds:

- The line bundle $\mathcal{O}(\Delta_i)$ is trivial on $U_{ij} \times U_{ik}$; and we fix once for all a trivialization.
- The line bundle M_σ is trivial on $U_{1,j} \times U_{2,k}$.

Let $\|\cdot\|_\ell$ be the metric on the line bundle $\mathcal{O}(P_\ell)_\sigma$. Let $\mathbb{I}_\ell$ be the canonical section of $\mathcal{O}(P_\ell)_\sigma$. There is a $\mathcal{C}^\infty$ function $\rho_{\ell ij}$ on U_{ij} such that

$$\|\mathbb{I}_\ell\|_\ell(z) = \rho_{\ell,ij}(z) \cdot \big|z - z(P_\ell)\big|.$$

Due to our choices, we can find (and fix once for all) two constants A_1 and A_2 *independent on the* P_i such that

$$A_1 \leqslant \rho_{\ell,ij}(z) \leqslant A_2.$$

Thus, we apply 3.7 and we find an absolute constant C_1, independent on P and on the d_i, such that

$$\sup\big\{\|\tilde{f}\|_\sigma\big\} \leqslant A \cdot C_1^{(d_1+d_2)}.$$

The section $\tilde{f}$ extends to a section, denote it again by $\tilde{f}$, of $(M \otimes \omega^{i_1}_{\mathcal{X}_1} \otimes \omega^{i_2}_{\mathcal{X}_2})_\sigma$ on a neighborhood of P, which we may suppose to be one of the products of the U_i above; a similar argument shows that $\sup\{\|\tilde{f}\|\} \leqslant A \cdot C_1^{(d_1+d_2)}$.

Let $\tilde{\mathcal{X}} \to \mathcal{X}_1 \times \mathcal{X}_2$ be the blow-up along the ideal $\mathcal{I}_{\underline{\vartheta},\delta-\epsilon,\underline{d}}$ and $E_{\delta-\epsilon}$ be the exceptional divisor; let $\tilde{P} : \mathrm{Spec}(O_K) \to \tilde{X}$ be the strict transform of P. By definition $\tilde{f}$ will give a non-zero section (which we will denote with the same symbol) $\tilde{f} \in H^0(\tilde{P}, M \otimes \omega^{i_1}_{\mathcal{X}_1/B} \otimes \omega^{i_2}_{\mathcal{X}_2/B}(-E_{\delta-\epsilon}))$. We will now give an upper bound for the norm of $\tilde{f}$. As before, once we take a suitably chosen (once for all) open covering of $(\mathcal{X}_i)_\sigma$, in the analytic topology, the existence of the upper bound as in the statement of the theorem is consequence of 5.2 below. $\square$

Let $\mathbb{D}$ be an open disk, $0 \in \mathbb{D}$ be a point on it and z be a coordinate with a simple zero on 0. Suppose that $\rho_i(z)$ $(i = 1, 2)$ are two $\mathcal{C}^\infty$ functions on $\mathbb{D}$; suppose that we can find two positive constants B_1 and B_2 such that $B_1 \leqslant \rho_i(z) \leqslant B_2$. We define two metrics $\|\cdot\|_i$ on $\mathcal{O}(0)$ by the formula $\|\mathbb{I}_0\|_i = |z|\rho_i(z)$.

Let $p_i : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ the ith projection, we will denote by $\mathcal{O}(-0_i)$ the line bundle $p_i^*(\mathcal{O}(-0))$ and by z_i the holomorphic function $p_i(z)$ (it is the canonical section of $\mathcal{O}(0_i)$). We will suppose that $\mathcal{O}(0_i)$ is equipped with the pull-back, via p_i of the metric $\|\cdot\|_i$.

Fix positive rational numbers ϑ_i and δ. For every couple of sufficiently divisible positive integers (d_1, d_2) define $\mathcal{I}_{\underline{\vartheta},\delta,\underline{d}}$ to be the ideal sheaf of $\mathcal{O}_{\mathbb{D}\times\mathbb{D}}$ generated by the monomials $z_1^{i_1} \cdot z_2^{i_2}$ with $\frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \geqslant \delta$.

Let $b : \tilde{X} \to \mathbb{D} \times \mathbb{D}$ be the blow-up of $\mathcal{I}_{\underline{\vartheta},\delta,\underline{d}}$ and $E := E_\delta \subset \tilde{X}$ be the exceptional divisor. In the same way as before, if the d_i are sufficiently big, we have a surjection

$$\bigoplus_{(i_1,i_2)\in H} \mathcal{O}(-i_1\cdot 0_1)\otimes\mathcal{O}(-i_2\cdot 0_2)\twoheadrightarrow \mathcal{I}_{\underline{\vartheta},\delta,\underline{d}}.$$

which induces a metric on $\mathcal{O}(E)$.

5.2. Theorem. *There exists a constant B depending only on ϑ_i, δ and the constants A_i such that if the d_i's are sufficiently big and divisible, $f\in H^0(\mathbb{D}\times\mathbb{D},\mathcal{I}_{\underline{\vartheta},\delta,\underline{d}})$ and $\tilde f$ is the corresponding section in $H^0(\tilde X,\mathcal{O}(-E))$ then, for every $z\in\tilde X$,*

$$\|\tilde f\|(z)\leqslant \|f\|\big(b(z)\big)\cdot B^{(d_1+d_2)}.$$

Proof. Denoting by $\mathbf{P}$ the projective bundle $\mathbf{Proj}(\bigoplus_{(i_1,i_2)\in H}\mathcal{O}(-i_1\cdot 0_1)\otimes\mathcal{O}(-i_2\cdot 0_2))$ over $\mathbb{D}\times\mathbb{D}$ we get a commutative diagram

$$\begin{tikzcd} \tilde X \arrow[r, hook, "\iota"] \arrow[dr] & \mathbf{P} \arrow[d] \\ & \mathbb{D}\times\mathbb{D}. \end{tikzcd}$$

Moreover, by construction we have an *isometry* $\iota^*(\mathcal{O}(1))\simeq\mathcal{O}(-E)$. Remark that $\mathbf{P}$ is isomorphic to $\mathbb{D}\times\mathbb{D}\times\mathbb{P}^N$ for a suitable N. Denote by $[u_{i_1,i_2}]_{(i_1,i_2)\in H}$ the homogeneous coordinates on $\mathbb{P}^N$; the blow-up $\tilde X$ is defined by the equations

$$u_{j_1,j_2}\cdot z_1^{i_1}\cdot z_2^{i_2}=u_{i_1,i_2}\cdot z_1^{j_1}\cdot z_2^{j_2}$$

for all (i_1,i_2) and (j_1,j_2) in H.

Let us work on the local chart $u_{i_1,i_2}\neq 0$; a local computation shows that over this chart

$$\|E\|=\frac{|z_1^{i_1}\cdot z_2^{i_2}|}{|u_{i_1,i_2}|}\cdot\sqrt{\sum_{(j_1,j_2)\in H}\big(|u_{j_1,j_2}|\cdot\rho_1^{j_1}\cdot\rho_2^{j_2}\big)^2}. \tag{5.3.1}$$

Let $f\in H^0(\mathbb{D}\times\mathbb{D},I_{\underline{\vartheta},\delta,\underline{d}})$. The pull-back $b^*(f)$ naturally defines a global section $\tilde f\in H^0(\tilde X,\mathcal{O}(-E))$. Over the chart $u_{i_1,i_2}\neq 0$ we can find a holomorphic function h such that $f=z_1^{i_1}\cdot z_2^{i_2}\cdot h$. In order to conclude the proof of the theorem we have to give an upper bound for

$$|h|\cdot\frac{\sqrt{\sum|u_{j_1,j_2}|^2\cdot\rho_1^{2j_1}\cdot\rho_2^{2j_2}}}{|u_{i_1,i_2}|}. \tag{5.4.1}$$

Fix a very small positive ϵ; we may suppose that we are in the disk

$$\frac{|u_{j_1,j_2}|}{|u_{i_1,i_2}|}\leqslant 1+\epsilon;$$

if this is not verified, it suffices to change the local chart. consequently, we can find a constant B_1 depending only on the norms (in particular independent on the d_i's) for which the expression in (5.4.1) is bounded from above by

$$|h|\cdot B_1^{(d_1+d_2)}.$$

Since h is holomorphic, the function $|h|$ will assume its maximum on the border. We may assume that the d_i are such that $\frac{d_i\cdot\delta}{\vartheta_i}\in\mathbb{N}$. On our chart, $z_1^{\frac{\delta\cdot d_1}{\vartheta_1}}=z_1^{i_1}\cdot z_2^{i_2}\cdot u_{\frac{\delta\cdot d_1}{\vartheta_1},0}$ (resp. $z_2^{\frac{\delta\cdot d_2}{\vartheta_2}}=z_1^{i_1}\cdot z_2^{i_2}\cdot u_{0,\frac{\delta\cdot d_1}{\vartheta_1}}$) and $|u_{\frac{\delta\cdot d_2}{\vartheta_2},0}|$ (resp. $u_{0,\frac{\delta\cdot d_2}{\vartheta_2}}$) is less or equal to $1+\epsilon$. Consequently, if $|z_1^{\frac{\delta\cdot d_1}{\vartheta_1}}|=1$ (resp. $|z_2^{\frac{\delta\cdot d_2}{\vartheta_2}}|=1$) then $1\leqslant|z_1^{i_1}\cdot z_2^{i_2}|\cdot(1+\epsilon)$ thus

$$|h|\leqslant\frac{\|f\|}{|z_1^{i_1}\cdot z_2^{i_2}|}\leqslant(1+\varepsilon)\cdot\|f\|;$$

the conclusion of the theorem easily follows. □

6. Proof of the main theorem

In this section we will give the proof of the main theorem of the paper: Theorem 2.2.

We recall all the tools and the ingredients: $\mathcal{X}_i$ are two regular arithmetic surfaces projective over $B:=\mathrm{Spec}(O_K)$ over which we fixed arithmetically ample hermitian line bundles M_i and symmetric hermitian metrics on $\mathcal{O}(\Delta_i)$ (Δ_i being the diagonal on $\mathcal{X}_i\times\mathcal{X}_i$). Eventually we fix a place $\sigma\in M_K$.

We fix two finite extensions L_i of K and two reduced divisors $D_i: B_{L_i}:=\mathrm{Spec}(O_{L_i})\to\mathcal{X}_i$. We denote by L the composite field $L_1\cdot L_2$ and by n the degree of the extension L/K. We fix two positive rational numbers $\vartheta_i\geqslant 1$ and a positive ϵ such that $\vartheta_1\cdot\vartheta_2\geqslant 2n+\epsilon$. We will denote by $T(D_i)$ the positive real number introduced in Section 2. We fix a function $\varphi:S\to[0;1]$ such that $\sum_{v\in S}\varphi(v)=1$. Theorem 2.2 will be consequence of the following:

6.1. Theorem. *There exists a constant A depending only on the arithmetic surfaces $\mathcal{X}_i$, the ϑ_i, the ϵ, the hermitian line bundles M_i, the symmetric metrics on the diagonals $\mathcal{O}(\Delta_i)$, the set S and the function φ, for which the following holds.*

Let $D_i\subset\mathcal{X}_i$ be divisors as above, and $P_i\in\mathcal{X}_i(B)$ be two rational sections such that

(i) *for every place $v\in S$, we have that $\lambda_{D_1,v}(P_1)>\varphi(v)\cdot\vartheta_1\cdot h_{M_1}(P_1)$ and $\lambda_{D_2,v}(P_2)>\varphi(v)\cdot\vartheta_1\cdot h_{M_2}(P_2)$;*
(ii) *$h_{M_1}(P_1)\geqslant A\cdot T(D_1)\cdot T(D_2)$.*

Then

$$h_{M_2}(P_2)\leqslant A\cdot T(D_1)\cdot T(D_2)\cdot h_{M_1}(P_1).$$

Proof. We first treat the case when, for at least one place, each of the P_i's is "far from D_i." Suppose that $v\in S$ is an infinite place, then take a covering of $(\mathcal{X}_i)_v$ by open sets U_{ij}, analytically equivalent to the disk of radius 1 and such that the open subsets analytically equivalent to the disk of radius 1/2 also cover the $(\mathcal{X}_i)_v$. We can then find a constant A_2 such that if U_{ij_k} are the open sets containing the $(D_i)_v$ and $(P_i)_v$ are not contained in the U_{ij_k} then $\lambda_{D_i,v}(P_i)\leqslant A_2$. Consequently, we see that, taking A much bigger then A_2 (which is independent on the D_i), in this case conditions (i) and (ii) are in contradiction. In particular the theorem holds in this case. A similar argument holds if v is a finite place.

Suppose that $\vartheta_1\cdot\vartheta_2=2n+\epsilon$; define $\epsilon_1:=\frac{\epsilon}{n+1}$ and $\delta:=2+\epsilon_0$. A suitable choice of ϵ_0 allows to suppose that the hypotheses of Proposition 3.6 are verified.

Here again, "absolute constant" will be equivalent to say "a constant which depends only on the $\mathcal{X}_i$, the hermitian line bundles M_i, the metrics on the diagonals, the ϑ_i's, but independent on the D_i's and on the d_i's."

For every couple of positive integers d_1 and d_2, let $\mathcal{I}_{\underline{\vartheta},\delta,\underline{d}}$ be the ideal sheaf on $\mathcal{X}_1\times\mathcal{X}_2$ defined in Section 3 and having support on $D_1\times D_2\subset\mathcal{X}_1\times\mathcal{X}_2$.

Fix an absolute constant A_3 such that $h_{\omega_{\mathcal{X}_i/B}}(\cdot)\leqslant A_3\cdot h_{M_i}(\cdot)$ and let ϵ_2 such that $\epsilon_2<\frac{\epsilon_1}{1+2\cdot A_3}$.

We apply 3.6 and we find an absolute constant A_4 such that, each time d_i's are sufficiently big and divisible we can find a non-zero global section $f \in H^0(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} \otimes M_2^{d_2} \otimes \mathcal{I}_{\underline{\vartheta},\delta,\underline{d}})$ such that

$$\sup_{\sigma \in M_\infty} \{\log \|f\|_\sigma\} \leqslant A_4 \cdot T(D_1) \cdot T(D_2)(d_1 + d_2).$$

One apply Theorem 4.2 with $\log(C) = A_3 \cdot T(D_1) \cdot T(D_2)$ and $\epsilon = \epsilon_2$ and deduce the existence of a constant A_5 for which, if a point P verify (a), (b) and (c) of Theorem 4.2 then the index $\mathrm{ind}_P(f, d_1, d_2) < \epsilon_2$; by Remark 4.5 one can see that A_5 is again of the form $A_6 \cdot T(D_1) \cdot T(D_2)$ with A_6 independent on the D_i's.

Suppose that $P_i : B \to \mathcal{X}_i$ are two sections which satisfy hypothesis (i) and such that

$$h_{M_2}(P_2) > A_6 \cdot T(D_1) \cdot T(D_2) h_{M_1}(P_1)$$

we will prove that there exists a constant A_7 such that $h_{M_1}(P_1) \leqslant A_7 \cdot T(D_1) \cdot T(D_2)$, and this will be the conclusion of the proof. $\square$

In the sequel we will denote by h_i the real numbers $h_{M_i}(P_i)$.

Take d to be a very big and divisible positive integer; let d_i be integers such that $d_i h_i \sim d$ and such that

$$\frac{h_2}{h_1} > \frac{d_1}{d_2}$$

(in order to keep the proof as readable as possible we avoid to introduce more small constants).

Let f be the section whose existence is assured by Proposition 3.6.

The hypotheses of Theorem 4.2 are satisfied consequently the index of f at $P_1 \times P_2$ is smaller then ϵ_2. Let $\tilde{\mathcal{X}} \to \mathcal{X}_1 \times \mathcal{X}_2$ be the blow-up of the ideal $\mathcal{I}_{\underline{\vartheta},\delta-\epsilon_2,\underline{d}}$ and $E_{\delta-\epsilon_2}$ (notations as in Section 5) be the exceptional divisor; let $\tilde{P} : B \to \tilde{\mathcal{X}}$ be the strict transform of $P := P_1 \times P_2$. We apply Theorem 5.1 and deduce the existence of an absolute constant A_8, a couple of indices (i_1, i_2) and a non-zero section $\tilde{f} \in H^0(\tilde{P}, M_1^{d_1} \otimes M_2^{d_2} \otimes \omega^{i_1}_{\mathcal{X}_1/B}|_{P_1} \otimes \omega^{i_2}_{\mathcal{X}_2/B}(-E_{\delta-\epsilon_2}))$ such that $\frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \leqslant \epsilon_2$ and $\sup_{\sigma \in M_\infty} \{\log \|\tilde{f}\|_\sigma\} \leqslant A_8 \cdot T(D_1) \cdot T(D_2)(d_1 + d_2)$.

6.2. Lemma. *Let $v \in M_K$. Then there is a positive constant C depending only on v (and on the $\mathcal{X}_i$'s, but independent on the D_i's, P_i's, d_i's) and a couple $(j_1^v; j_2^v)$ such that $\frac{j_1^v}{d_1} \cdot \vartheta_1 + \frac{j_2^v}{d_2} \geqslant \delta - \epsilon_2$ and*

$$-\log \|E_{\delta-\epsilon_2}\|_v(\tilde{P}) \geqslant j_1^v \lambda_{D_1,v}(P_1) + j_2^v \lambda_{D_2,v}(P_2) - C(d_1 + d_2).$$

Proof. We prove the case when $v \in M_\infty$; when v is finite the proof is similar (and even easier).

We use the results and the notations of the proof of Theorem 5.2.

We may suppose that there is a couple $(j_1^v; j_2^v)$ with $\frac{j_1^v}{d_1} \cdot \vartheta_1 + \frac{j_2^v}{d_2} \geqslant \delta - \epsilon_2$ such that $\tilde{P}_v$ belongs to the open set $|u_{j_1,j_2}| < (1+\epsilon)|u_{j_1^v,j_2^v}|$ for every (j_1, j_2) in H. From the formula (5.3.1)

$$\|E_{\delta-\epsilon_2}\| = |z_1^{j_1^v} \cdot z_2^{j_2^v}| \cdot \sqrt{\sum_{(j_1,j_2) \in H} \left| \frac{|u_{j_1,j_2}| \cdot \rho_1^{j_1} \cdot \rho_2^{j_2}}{|u_{i_1,i_2}|} \right|^2}.$$

Thus

$$\log \|E_{\delta-\epsilon_2}\| \leqslant j_1^v \log|z_1| + j_2^v \log|z_2| + C(d_1 + d_2).$$

The conclusion follows from the properties of the Weil functions.

From the lemma above, we deduce the existence of a constant C such that

$$\widehat{\deg}\big(\tilde{P}^*(E_{\delta-\epsilon_2})\big) \geqslant \sum_{v\in S} j_1^v \lambda_{D_1,v}(P_1) + j_2^v \lambda_{D_2,v}(P_2) - C(d_1+d_2).$$

Consequently, from the hypotheses

$$\begin{aligned}\widehat{\deg}\big(\tilde{P}^*(E_{\delta-\epsilon_2})\big) &\geqslant \sum_{v\in S} j_1^v \cdot \varphi(v)\cdot \vartheta_1 \cdot h_1 + j_2^v \cdot \varphi(v)\cdot \vartheta_2 \cdot d_2 \cdot h_2 - C(d_1+d_2)\\ &= \sum_{v\in S}\left(\frac{j_1^v}{d_1}\cdot \vartheta_1 \cdot d_1 h_1 + \frac{j_2^v}{d_2}\cdot \vartheta_2 \cdot h_2\right)\cdot \varphi(v) - C(d_1+d_2)\\ &\geqslant d(2+\epsilon_1-\epsilon_2) - C(d_1+d_2).\end{aligned}$$

Thus we deduce

$$\begin{aligned}&-A_8 \cdot T(D_1)\cdot T(D_2)(d_1+d_2)\\ &\quad\leqslant d_1\cdot h_1 + d_2 \cdot h_2 + i_1 \cdot h_{\omega_{\mathcal{X}_1/B}}(P_1) + i_2 \cdot h_{\omega_{\mathcal{X}_2/B}}(P_2) - \widehat{\deg}\big(\tilde{P}^*(E_{\delta-\epsilon_2})\big)\\ &\quad\leqslant \big((2+2\epsilon_2\cdot A_3) - (2+\epsilon_1-\epsilon_2)\big)\cdot d + C(d_1+d_2).\end{aligned}$$

From this and by our choice of the ϵ_i's, we deduce the existence of a constant A_9 such that

$$-A_9 \cdot T(D_1)\cdot T(D_2)\cdot\left(\frac{1}{h_1}+\frac{1}{h_2}\right) \leqslant -\epsilon_3,$$

where $\epsilon_3 = \epsilon_1 - (1+2A_3)\cdot\epsilon_2$; thus

$$h_1 \leqslant \frac{2\cdot A_9}{\epsilon_3}\cdot T(D_1)\cdot T(D_2)$$

and from this the conclusion follows. □

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